THE 1991 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Question 1
Let $G$ be the centroid of triangle $ABC$ and $M$ be the midpoint of $BC$. Let $X$ be on $AB$ and $Y$ on $AC$ such that the points $X$, $Y$, and $G$ are collinear and $XY$ and $BC$ are parallel. Suppose that $XC$ and $GB$ intersect at $Q$ and $YB$ and $GC$ intersect at $P$. Show that triangle $MPQ$ is similar to triangle $ABC$.

Question 2
Suppose there are 997 points given in a plane. If every two points are joined by a line segment with its midpoint coloured in red, show that there are at least 1991 red points in the plane. Can you find a special case with exactly 1991 red points?

Question 3
Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$. Show that

$$\frac{a_1^2}{a_1 + b_1} + \frac{a_2^2}{a_2 + b_2} + \cdots + \frac{a_n^2}{a_n + b_n} \geq \frac{a_1 + a_2 + \cdots + a_n}{2}.$$ 

Question 4
During a break, $n$ children at school sit in a circle around their teacher to play a game. The teacher walks clockwise close to the children and hands out candies to some of them according to the following rule. He selects one child and gives him a candy, then he skips the next child and gives a candy to the next one, then he skips 2 and gives a candy to the next one, then he skips 3, and so on. Determine the values of $n$ for which eventually, perhaps after many rounds, all children will have at least one candy each.

Question 5
Given are two tangent circles and a point $P$ on their common tangent perpendicular to the lines joining their centres. Construct with ruler and compass all the circles that are tangent to these two circles and pass through the point $P$. 