THE 1993 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Question 1
Let $ABCD$ be a quadrilateral such that all sides have equal length and angle $ABC$ is 60 deg. Let $l$ be a line passing through $D$ and not intersecting the quadrilateral (except at $D$). Let $E$ and $F$ be the points of intersection of $l$ with $AB$ and $BC$ respectively. Let $M$ be the point of intersection of $CE$ and $AF$.
Prove that $CA^2 = CM \times CE$.

Question 2
Find the total number of different integer values the function
\[ f(x) = \lfloor x \rfloor + \lfloor 2x \rfloor + \left\lfloor \frac{5x}{3} \right\rfloor + \lfloor 3x \rfloor + \lfloor 4x \rfloor \]
takes for real numbers $x$ with $0 \leq x \leq 100$.

Question 3
Let
\[ f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \quad \text{and} \quad g(x) = c_{n+1} x^{n+1} + c_n x^n + \cdots + c_0 \]
be non-zero polynomials with real coefficients such that $g(x) = (x + r) f(x)$ for some real number $r$. If $a = \max(|a_n|, \ldots, |a_0|)$ and $c = \max(|c_{n+1}|, \ldots, |c_0|)$, prove that $\frac{a}{c} \leq n + 1$.

Question 4
Determine all positive integers $n$ for which the equation
\[ x^n + (2 + x)^n + (2 - x)^n = 0 \]
has an integer as a solution.

Question 5
Let $P_1, P_2, \ldots, P_{1993} = P_0$ be distinct points in the $xy$-plane with the following properties:
(i) both coordinates of $P_i$ are integers, for $i = 1, 2, \ldots, 1993$;
(ii) there is no point other than $P_i$ and $P_{i+1}$ on the line segment joining $P_i$ with $P_{i+1}$ whose coordinates are both integers, for $i = 0, 1, \ldots, 1992$.
Prove that for some $i$, $0 \leq i \leq 1992$, there exists a point $Q$ with coordinates $(q_x, q_y)$ on the line segment joining $P_i$ with $P_{i+1}$ such that both $2q_x$ and $2q_y$ are odd integers.