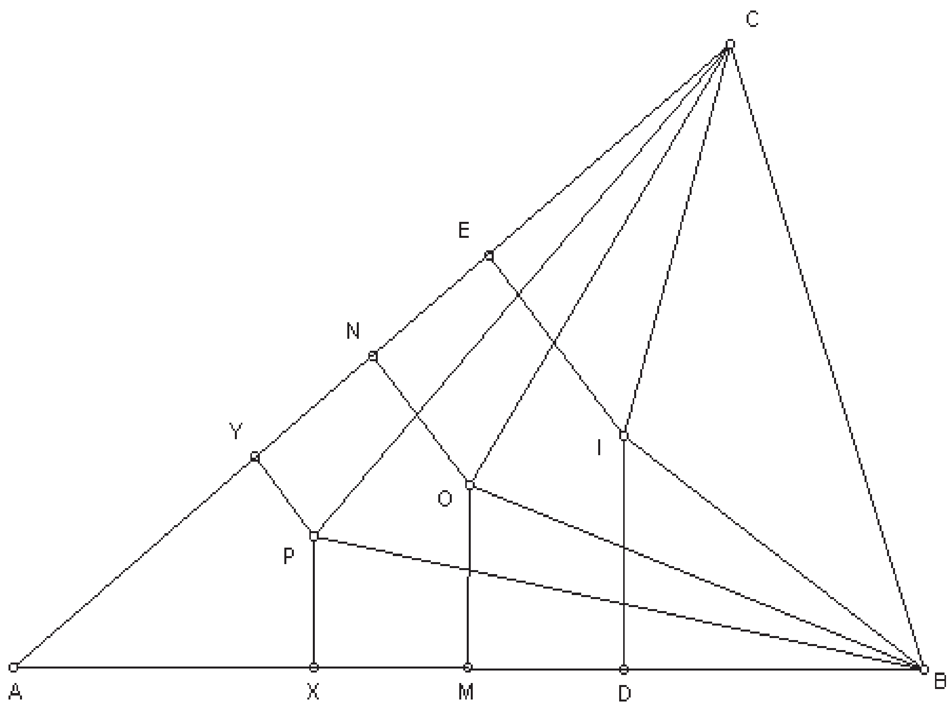
XX Asian Pacific Mathematics Olympiad

March, 2008

Problem 1. Let ABC be a triangle with $\angle A < 60^\circ$. Let X and Y be the points on the sides AB and AC, respectively, such that $CA + AX = CB + BX$ and $BA + AY = BC + CY$. Let P be the point in the plane such that the lines PX and PY are perpendicular to AB and AC, respectively. Prove that $\angle BPC < 120^\circ$.

(Solution) Let I be the incenter of $\triangle ABC$, and let the feet of the perpendiculars from I to AB and to AC be D and E, respectively. (Without loss of generality, we may assume that AC is the longest side. Then X lies on the line segment AD. Although P may or may not lie inside $\triangle ABC$, the proof below works for both cases. Note that P is on the line perpendicular to AB passing through X.) Let O be the midpoint of IP, and let the feet of the perpendiculars from O to AB and to AC be M and N, respectively. Then M and N are the midpoints of DX and EY, respectively.



The conditions on the points X and Y yield the equations

$$AX = \frac{AB + BC - CA}{2} \qquad \text{and} \qquad AY = \frac{BC + CA - AB}{2}.$$

From $AD = AE = \dfrac{CA + AB - BC}{2}$, we obtain

$$BD = AB - AD = AB - \frac{CA + AB - BC}{2} = \frac{AB + BC - CA}{2} = AX.$$

Since M is the midpoint of DX, it follows that M is the midpoint of AB. Similarly, N is the midpoint of AC. Therefore, the perpendicular bisectors of AB and AC meet at O, that is, O is the circumcenter of $\triangle ABC$. Since $\angle BAC < 60^\circ$, O lies on the same side of BC as the point A and

$$\angle BOC = 2\angle BAC.$$

We can compute $\angle BIC$ as follows:

$$\begin{aligned}\angle BIC &= 180^\circ - \angle IBC - \angle ICB = 180^\circ - \frac{1}{2}\angle ABC - \frac{1}{2}\angle ACB \\ &= 180^\circ - \frac{1}{2}(\angle ABC + \angle ACB) = 180^\circ - \frac{1}{2}(180^\circ - \angle BAC) = 90^\circ + \frac{1}{2}\angle BAC\end{aligned}$$

It follows from $\angle BAC < 60^\circ$ that

$$2\angle BAC < 90^\circ + \frac{1}{2}\angle BAC, \quad \text{i.e.,} \quad \angle BOC < \angle BIC.$$

From this it follows that I lies inside the circumcircle of the isosceles triangle BOC because O and I lie on the same side of BC. However, as O is the midpoint of IP, P must lie outside the circumcircle of triangle BOC and on the same side of BC as O. Therefore

$$\angle BPC < \angle BOC = 2\angle BAC < 120^\circ.$$

Remark. If one assumes that $\angle A$ is smaller than the other two, then it is clear that the line PX (or the line perpendicular to AB at X if $P = X$) runs through the excenter I_C of the excircle tangent to the side AB. Since $2\angle ACI_C = \angle ACB$ and $BC < AC$, we have $2\angle PCB > \angle C$. Similarly, $2\angle PBC > \angle B$. Therefore,

$$\angle BPC = 180^\circ - (\angle PBC + \angle PCB) < 180^\circ - \left(\frac{\angle B + \angle C}{2}\right) = 90 + \frac{\angle A}{2} < 120^\circ.$$

In this way, a special case of the problem can be easily proved.

Problem 2. Students in a class form groups each of which contains exactly three members such that any two distinct groups have at most one member in common. Prove that, when the class size is 46, there is a set of 10 students in which no group is properly contained.

(Solution) We let C be the set of all 46 students in the class and let

$$s := \max\{\, |S| \,:\, S \subseteq C \text{ such that } S \text{ contains no group properly}\,\}.$$

Then it suffices to prove that $s \geq 10$. (If $|S| = s > 10$, we may choose a subset of S consisting of 10 students.)

Suppose that $s \leq 9$ and let S be a set of size s in which no group is properly contained. Take any student, say v, from outside S. Because of the maximality of s, there should be a group containing the student v and two other students in S. The number of ways to choose two students from S is

$$\binom{s}{2} \leq \binom{9}{2} = 36.$$

On the other hand, there are at least $37 = 46 - 9$ students outside of S. Thus, among those 37 students outside, there is at least one student, say u, who does not belong to any group containing two students in S and one outside. This is because no two distinct groups have two members in common. But then, S can be enlarged by including u, which is a contradiction.

Remark. One may choose a subset S of C that contains no group properly. Then, assuming $|S| < 10$, prove that there is a student outside S, say u, who does not belong to any group containing two students in S. After enlarging S by including u, prove that the enlarged S still contains no group properly.

Problem 3. Let Γ be the circumcircle of a triangle ABC. A circle passing through points A and C meets the sides BC and BA at D and E, respectively. The lines AD and CE meet Γ again at G and H, respectively. The tangent lines of Γ at A and C meet the line DE at L and M, respectively. Prove that the lines LH and MG meet at Γ.

(Solution) Let MG meet Γ at P. Since $\angle MCD = \angle CAE$ and $\angle MDC = \angle CAE$, we have $MC = MD$. Thus

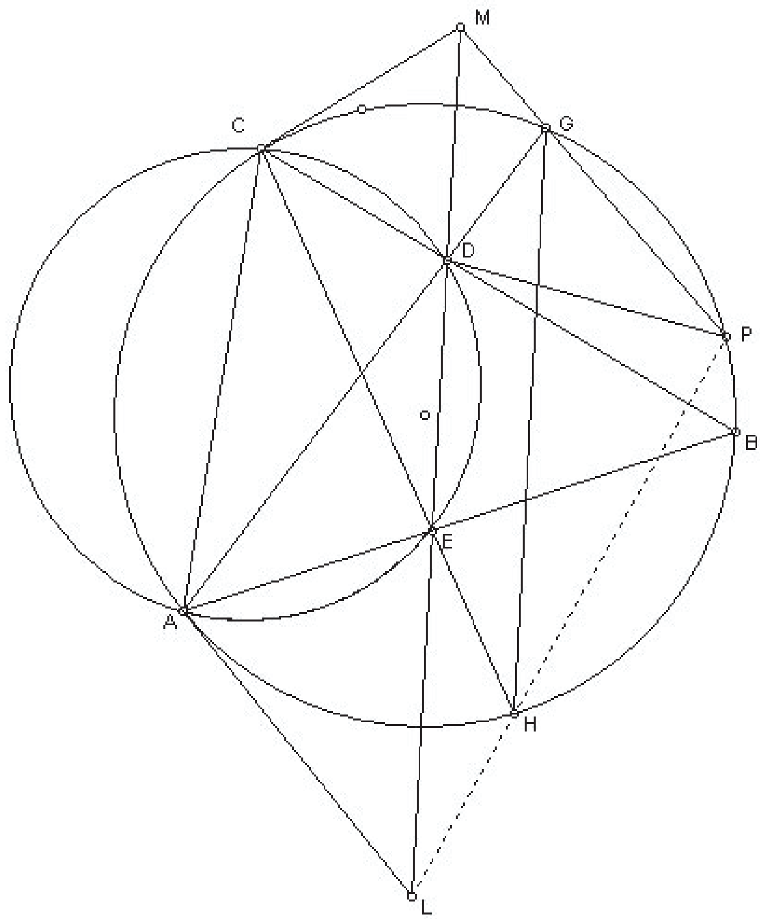
$$MD^2 = MC^2 = MG \cdot MP$$

and hence MD is tangent to the circumcircle of $\triangle DGP$. Therefore $\angle DGP = \angle EDP$.

Let Γ' be the circumcircle of $\triangle BDE$. If $B = P$, then, since $\angle BGD = \angle BDE$, the tangent lines of Γ' and Γ at B should coincide, that is Γ' is tangent to Γ from inside. Let $B \neq P$. If P lies in the same side of the line BC as G, then we have

$$\angle EDP + \angle ABP = 180^\circ$$

because $\angle DGP + \angle ABP = 180^\circ$. That is, the quadrilateral $BPDE$ is cyclic, and hence P is on the intersection of Γ' with Γ.



Otherwise,

$$\angle EDP = \angle DGP = \angle AGP = \angle ABP = \angle EBP.$$

Therefore the quadrilateral $PBDE$ is cyclic, and hence P again is on the intersection of Γ' with Γ.

Similarly, if LH meets Γ at Q, we either have $Q = B$, in which case Γ' is tangent to Γ from inside, or $Q \neq B$. In the latter case, Q is on the intersection of Γ' with Γ. In either case, we have $P = Q$.

Problem 4. Consider the function $f : \mathbb{N}_0 \to \mathbb{N}_0$, where $\mathbb{N}_0$ is the set of all non-negative integers, defined by the following conditions:

$$\text{(i) } f(0) = 0, \text{ (ii) } f(2n) = 2f(n) \text{ and (iii) } f(2n+1) = n + 2f(n) \text{ for all } n \geq 0.$$

(a) Determine the three sets $L := \{ n \mid f(n) < f(n+1) \}$, $E := \{ n \mid f(n) = f(n+1) \}$, and $G := \{ n \mid f(n) > f(n+1) \}$.

(b) For each $k \geq 0$, find a formula for $a_k := \max\{f(n) : 0 \leq n \leq 2^k\}$ in terms of k.

(Solution) (a) Let

$$L_1 := \{2k : k > 0\}, \quad E_1 := \{0\} \cup \{4k+1 : k \geq 0\}, \quad \text{and} \quad G_1 := \{4k+3 : k \geq 0\}.$$

We will show that $L_1 = L, E_1 = E$, and $G_1 = G$. It suffices to verify that $L_1 \subseteq E$, $E_1 \subseteq E$, and $G_1 \subseteq G$ because L_1, E_1, and G_1 are mutually disjoint and $L_1 \cup E_1 \cup G_1 = \mathbb{N}_0$.

Firstly, if $k > 0$, then $f(2k) - f(2k+1) = -k < 0$ and therefore $L_1 \subseteq L$.

Secondly, $f(0) = 0$ and

$$\begin{aligned} f(4k+1) &= 2k + 2f(2k) = 2k + 4f(k) \\ f(4k+2) &= 2f(2k+1) = 2(k + 2f(k)) = 2k + 4f(k) \end{aligned}$$

for all $k \geq 0$. Thus, $E_1 \subseteq E$.

Lastly, in order to prove $G_1 \subset G$, we claim that $f(n+1) - f(n) \leq n$ for all n. (In fact, one can prove a stronger inequality: $f(n+1) - f(n) \leq n/2$.) This is clearly true for even n from the definition since for $n = 2t$,

$$f(2t+1) - f(2t) = t \leq n.$$

If $n = 2t+1$ is odd, then (assuming inductively that the result holds for all nonnegative $m < n$), we have

$$\begin{aligned} f(n+1) - f(n) &= f(2t+2) - f(2t+1) = 2f(t+1) - t - 2f(t) \\ &= 2(f(t+1) - f(t)) - t \leq 2t - t = t < n. \end{aligned}$$

For all $k \geq 0$,

$$\begin{aligned} &f(4k+4) - f(4k+3) = f(2(2k+2)) - f(2(2k+1)+1) \\ &= 4f(k+1) - (2k+1+2f(2k+1)) = 4f(k+1) - (2k+1+2k+4f(k)) \\ &= 4(f(k+1) - f(k)) - (4k+1) \leq 4k - (4k+1) < 0. \end{aligned}$$

This proves $G_1 \subseteq G$.

(b) Note that $a_0 = a_1 = f(1) = 0$. Let $k \geq 2$ and let $N_k = \{0, 1, 2, \ldots, 2^k\}$. First we claim that the maximum a_k occurs at the largest number in $G \cap N_k$, that is, $a_k = f(2^k - 1)$. We use mathematical induction on k to prove the claim. Note that $a_2 = f(3) = f(2^2 - 1)$.

Now let $k \geq 3$. For every even number $2t$ with $2^{k-1} + 1 < 2t \leq 2^k$,

$$f(2t) = 2f(t) \leq 2a_{k-1} = 2f(2^{k-1} - 1) \tag{†}$$

by induction hypothesis. For every odd number $2t+1$ with $2^{k-1} + 1 \leq 2t + 1 < 2^k$,

$$\begin{aligned} f(2t+1) &= t + 2f(t) \leq 2^{k-1} - 1 + 2f(t) \\ &\leq 2^{k-1} - 1 + 2a_{k-1} = 2^{k-1} - 1 + 2f(2^{k-1} - 1) \end{aligned} \tag{‡}$$

again by induction hypothesis. Combining (†), (‡) and

$$f(2^k - 1) = f(2(2^{k-1} - 1) + 1) = 2^{k-1} - 1 + 2f(2^{k-1} - 1),$$

we may conclude that $a_k = f(2^k - 1)$ as desired.

Furthermore, we obtain

$$a_k = 2a_{k-1} + 2^{k-1} - 1$$

for all $k \geq 3$. Note that this recursive formula for a_k also holds for $k \geq 0, 1$ and 2. Unwinding this recursive formula, we finally get

$$\begin{aligned} a_k &= 2a_{k-1} + 2^{k-1} - 1 = 2(2a_{k-2} + 2^{k-2} - 1) + 2^{k-1} - 1 \\ &= 2^2 a_{k-2} + 2 \cdot 2^{k-1} - 2 - 1 = 2^2(2a_{k-3} + 2^{k-3} - 1) + 2 \cdot 2^{k-1} - 2 - 1 \\ &= 2^3 a_{k-3} + 3 \cdot 2^{k-1} - 2^2 - 2 - 1 \\ &\vdots \\ &= 2^k a_0 + k2^{k-1} - 2^{k-1} - 2^{k-2} - ... - 2 - 1 \\ &= k2^{k-1} - 2^k + 1 \quad \text{for all } k \geq 0. \end{aligned}$$

Problem 5. Let a, b, c be integers satisfying $0 < a < c-1$ and $1 < b < c$. For each k, $0 \le k \le a$, let r_k, $0 \le r_k < c$, be the remainder of kb when divided by c. Prove that the two sets $\{r_0, r_1, r_2, \ldots, r_a\}$ and $\{0, 1, 2, \ldots, a\}$ are different.

(Solution) Suppose that two sets are equal. Then $\gcd(b, c) = 1$ and the polynomial

$$f(x) := (1 + x^b + x^{2b} + \cdots + x^{ab}) - (1 + x + x^2 + \cdots + x^{a-1} + x^a)$$

is divisible by $x^c - 1$. (This is because: $m = n + cq \implies x^m - x^n = x^{n+cq} - x^n = x^n(x^{cq} - 1)$ and $(x^{cq} - 1) = (x^c - 1)((x^c)^{q-1} + (x^c)^{q-2} + \cdots + 1)$.) From

$$f(x) = \frac{x^{(a+1)b} - 1}{x^b - 1} - \frac{x^{a+1} - 1}{x - 1} = \frac{F(x)}{(x-1)(x^b - 1)} \ ,$$

where $F(x) = x^{ab+b+1} + x^b + x^{a+1} - x^{ab+b} - x^{a+b+1} - x$, we have

$$F(x) \equiv 0 \pmod{x^c - 1}.$$

Since $x^c \equiv 1 \pmod{x^c - 1}$, we may conclude that

$$\{ab + b + 1, b, a + 1\} \equiv \{ab + b, a + b + 1, 1\} \pmod{c}. \qquad (\dagger)$$

Thus,

$$b \equiv \ ab + b,\ a + b + 1 \ \text{ or } \ 1 \pmod{c}.$$

But neither $b \equiv 1 \pmod{c}$ nor $b \equiv a + b + 1 \pmod{c}$ are possible by the given conditions. Therefore, $b \equiv ab + b \pmod{c}$. But this is also impossible because $\gcd(b, c) = 1$.