SOLUTIONS FOR 2010 APMO PROBLEMS

Problem 1. Let ABC be a triangle with $\angle BAC \neq 90^\circ$. Let O be the circumcenter of the triangle ABC and let Γ be the circumcircle of the triangle BOC. Suppose that Γ intersects the line segment AB at P different from B, and the line segment AC at Q different from C. Let ON be a diameter of the circle Γ. Prove that the quadrilateral $APNQ$ is a parallelogram.

Solution: From the assumption that the circle Γ intersects both of the line segments AB and AC, it follows that the 4 points N, C, Q, O are located on Γ in the order of N, C, Q, O or in the order of N, C, O, Q. The following argument for the proof of the assertion of the problem is valid in either case. Since $\angle NQC$ and $\angle NOC$ are subtended by the same arc $\overset{\frown}{NC}$ of Γ at the points Q and O, respectively, on Γ, we have $\angle NQC = \angle NOC$. We also have $\angle BOC = 2\angle BAC$, since $\angle BOC$ and $\angle BAC$ are subtended by the same arc $\overset{\frown}{BC}$ of the circum-circle of the triangle ABC at the center O of the circle and at the point A on the circle, respectively. From $OB = OC$ and the fact that ON is a diameter of Γ, it follows that the triangles OBN and OCN are congruent, and therefore we obtain $2\angle NOC = \angle BOC$. Consequently, we have $\angle NQC = \frac{1}{2}\angle BOC = \angle BAC$, which shows that the 2 lines AP, QN are parallel.

In the same manner, we can show that the 2 lines AQ, PN are also parallel. Thus, the quadrilateral $APNQ$ is a parallelogram.

Problem 2. For a positive integer k, call an integer a *pure k-th power* if it can be represented as m^k for some integer m. Show that for every positive integer n there exist n distinct positive integers such that their sum is a pure 2009-th power, and their product is a pure 2010-th power.

Solution: For the sake of simplicity, let us set $k = 2009$.

First of all, choose n distinct positive integers $b_1, \cdots, b_n$ suitably so that their product is a pure $k+1$-th power (for example, let $b_i = i^{k+1}$ for $i = 1, \cdots, n$). Then we have $b_1 \cdots b_n = t^{k+1}$ for some positive integer t. Set $b_1 + \cdots + b_n = s$.

Now we set $a_i = b_i s^{k^2-1}$ for $i = 1, \cdots, n$, and show that $a_1, \cdots, a_n$ satisfy the required conditions. Since $b_1, \cdots, b_n$ are distinct positive integers, it is clear that so are $a_1, \cdots, a_n$. From

$$a_1 + \cdots + a_n = s^{k^2-1}(b_1 + \cdots + b_n) = s^{k^2} = (s^k)^{2009},$$
$$a_1 \cdots a_n = (s^{k^2-1})^n b_1 \cdots b_n = (s^{k^2-1})^n t^{k+1} = (s^{(k-1)n} t)^{2010}$$

we can see that $a_1, \cdots, a_n$ satisfy the conditions on the sum and the product as well. This ends the proof of the assertion.

Remark: We can find the appropriate exponent $k^2 - 1$ needed for the construction of the a_i's by solving the simultaneous congruence relations: $x \equiv 0 \pmod{k+1}$, $x \equiv -1 \pmod{k}$.

Problem 3. Let n be a positive integer. n people take part in a certain party. For any pair of the participants, either the two are acquainted with each other or they are not. What is the maximum possible number of the pairs for which the two are not acquainted but have a common acquaintance among the participants?

Solution: When 1 participant, say the person A, is mutually acquainted with each of the remaining $n-1$ participants, and if there are no other acquaintance relationships among the participants, then for any pair of participants not involving A, the two are not mutual acquaintances, but they have a common acquaintance, namely A, so any such pair satisfies the requirement. Thus, the number desired in this case is $\frac{(n-1)(n-2)}{2} = \frac{n^2-3n+2}{2}$.

Let us show that $\frac{n^2-3n+2}{2}$ is the maximum possible number of the pairs satisfying the requirement of the problem. First, let us observe that in the process of trying to find the maximum possible number of such pairs, if we split the participants into two non-empty subsets T and S which are disjoint, we may assume that there is a pair consisting of one person chosen from T and the other chosen from S who are mutual acquaintances. This is so, since if there are no such pair for some splitting T and S, then among the pairs consisting of one person chosen from T and the other chosen from S, there is no pair for which the two have a common acquaintance among participants, and therefore, if we arbitrarily choose a person $A \in T$ and $B \in S$ and declare that A and B are mutual acquaintances, the number of the pairs satisfying the requirement of the problem does not decrease.

Let us now call a set of participants a *group* if it satisfies the following 2 conditions:

• One can connect any person in the set with any other person in the set by tracing a chain of mutually acquainted pairs. More precisely, for any pair of people A, B in the set there exists a sequence of people $A_0, A_1, \cdots, A_n$ for which $A_0 = A$, $A_n = B$ and, for each $i : 0 \leq i \leq n-1$, A_i and A_{i+1} are mutual acquaintances.

• No person in this set can be connected with a person not belonging to this set by tracing a chain of mutually acquainted pairs.

In view of the discussions made above, we may assume that the set of all the participants to the party forms a group of n people. Let us next consider the following lemma.

Lemma. In a group of n people, there are at least $n-1$ pairs of mutual acquaintances.

Proof: If you choose a mutually acquainted pair in a group and declare the two in the pair are not mutually acquainted, then either the group stays the same or splits into 2 groups. This means that by changing the status of a mutually acquainted pair in a group to that of a non-acquainted pair, one can increase the number of groups at most by 1. Now if in a group of n people you change the status of all of the mutually acquainted pairs to that of non-acquainted pairs, then obviously, the number of groups increases from 1 to n. Therefore, there must be at least $n-1$ pairs of mutually acquainted pairs in a group consisting of n people. □

The lemma implies that there are at most $\frac{n(n-1)}{2} - (n-1) = \frac{n^2-3n+2}{2}$ pairs satisfying the condition of the problem. Thus the desired maximum number of pairs satisfying the requirement of the problem is $\frac{n^2-3n+2}{2}$.

Remark: One can give a somewhat different proof by separating into 2 cases depending on whether there are at least $n-1$ mutually acquainted pairs, or at most $n-2$ such pairs. In the former case, one can argue in the same way as the proof above, while in the latter case, the Lemma above implies that there would be 2 or more groups to start with, but then, in view of the comment made before the definition of a group above, these groups can be combined to form one group, thereby one can reduce the argument to the former case.

Alternate Solution 1: The construction of an example for the case for which the number $\frac{n^2-3n+2}{2}$ appears, and the argument for the case where there is only 1 group would be the same as in the preceding proof.

Suppose, then, n participants are separated into $k (k \geq 2)$ groups, and the number of people in each group is given by $a_i, i = 1, \cdots, k$. In such a case, the number of pairs for which paired people are not mutually acquainted but have a common acquaintance is at most $\sum_{i=1}^{k} {}_{a_i}C_2$, where we set ${}_1C_2 = 0$ for convenience. Since ${}_aC_2 + {}_bC_2 \leq {}_{a+b}C_2$ holds for any pair of positive integers a, b, we have $\sum_{i=1}^{k} {}_{a_i}C_2 \leq {}_{a_1}C_2 + {}_{n-a_1}C_2$. From

$$ {}_{a_1}C_2 + {}_{n-a_1}C_2 = a_1^2 - na_1 + \frac{n^2-n}{2} = \left(a_1 - \frac{n}{2}\right)^2 + \frac{n^2-2n}{4} $$

it follows that ${}_{a_1}C_2 + {}_{n-a_1}C_2$ takes its maximum value when $a_1 = 1, n-1$. Therefore, we have $\sum_{i=1}^{k} {}_{a_i}C_2 \leq {}_{n-1}C_2$, which shows that in the case where the number of groups are 2 or more, the number of the pairs for which paired people are not mutually acquainted but have a common acquaintance is at most ${}_{n-1}C_2 = \frac{n^2-3n+2}{2}$, and hence the desired maximum number of the pairs satisfying the requirement is $\frac{n^2-3n+2}{2}$.

Alternate Solution 2: Construction of an example would be the same as the preceding proof.

For a participant, say A, call another participant, say B, a *familiar face* if A and B are not mutually acquainted but they have a common acquaintance among the participants, and in this case call the pair A, B a familiar pair.

Suppose there is a participant P who is mutually acquainted with d participants. Denote by S the set of these d participants, and by T the set of participants different from P and not belonging to the set S. Suppose there are e pairs formed by a person in S and a person in T who are mutually acquainted.

Then the number of participants who are familiar faces to P is at most e. The number of pairs formed by two people belonging to the set S and are mutually acquainted is at most ${}_dC_2$. The number of familiar pairs formed by two people belonging to the set T is at most ${}_{n-d-1}C_2$. Since there are e pairs formed by a person in the set S and a person in the set T who are mutually acquainted (and so the pairs are not familiar pairs), we have at most $d(n-1-d)-e$ familiar pairs formed by a person chosen from S and a person chosen from T. Putting these together we conclude that there are at most $e + {}_dC_2 + {}_{n-1-d}C_2 + d(n-1-d) - e$ familiar pairs. Since

$$ e + {}_dC_2 + {}_{n-1-d}C_2 + d(n-1-d) - e = \frac{n^2-3n+2}{2}, $$

the number we seek is at most $\frac{n^2-3n+2}{2}$, and hence this is the desired solution to the problem.

Problem 4. Let ABC be an acute triangle satisfying the condition $AB > BC$ and $AC > BC$. Denote by O and H the circumcenter and the orthocenter, respectively, of the triangle ABC. Suppose that the circumcircle of the triangle AHC intersects the line AB at M different from A, and that the circumcircle of the triangle AHB intersects the line AC at N different from A. Prove that the circumcenter of the triangle MNH lies on the line OH.

Solution: In the sequel, we denote $\angle BAC = \alpha, \angle CBA = \beta, \angle ACB = \gamma$. Let O' be the circumcenter of the triangle MNH. The lengths of line segments starting from the point H will be treated as signed quantities.

Let us denote by M', N' the point of intersection of CH, BH, respectively, with the circumcircle of the triangle ABC (distinct from C, B, respectively.) From the fact that 4 points A, M, H, C lie on the same circle, we see that $\angle MHM' = \alpha$ holds. Furthermore, $\angle BM'C, \angle BN'C$ and α are all subtended by the same arc $\overset{\frown}{BC}$ of the circumcircle of the triangle ABC at points on the circle, and therefore, we have $\angle BM'C = \alpha$, and $\angle BN'C = \alpha$ as well. We also have $\angle ABH = \angle ACN'$ as they are subtended by the same arc $\overset{\frown}{AN'}$ of the circumcircle of the triangle ABC at points on the circle. Since $HM' \perp BM, HN' \perp AC$, we conclude that

$$\angle M'HB = 90^\circ - \angle ABH = 90^\circ - \angle ACN' = \alpha$$

is valid as well. Putting these facts together, we obtain the fact that the quadrilateral $HBM'M$ is a rhombus. In a similar manner, we can conclude that the quadrilateral $HCN'N$ is also a rhombus. Since both of these rhombuses are made up of 4 right triangles with an angle of magnitude α, we also see that these rhombuses are similar.

Let us denote by P, Q the feet of the perpendicular lines on HM and HN, respectively, drawn from the point O'. Since O' is the circumcenter of the triangle MNH, P, Q are respectively, the midpoints of the line segments HM, HN. Furthermore, if we denote by R, S the feet of the perpendicular lines on HM and HN, respectively, drawn from the point O, then since O is the circumcenter of both the triangle $M'BC$ and the triangle $N'BC$, we see that R is the intersection point of HM and the perpendicular bisector of BM', and S is the intersection point of HN and the perpendicular bisector of CN'.

We note that the similarity map ϕ between the rhombuses $HBM'M$ and $HCN'N$ carries the perpendicular bisector of BM' onto the perpendicular bisector of CN', and straight line HM onto the straight line HN, and hence ϕ maps R onto S, and P onto Q. Therefore, we get $HP : HR = HQ : HS$. If we now denote by X, Y the intersection points of the line HO' with the line through R and perpendicular to HP, and with the line through S and perpendicular to HQ, respectively, then we get

$$HO' : HX = HP : HR = HQ : HS = HO' : HY$$

so that we must have $HX = HY$, and therefore, $X = Y$. But it is obvious that the point of intersection of the line through R and perpendicular to HP with the line through S and perpendicular to HQ must be O, and therefore, we conclude that $X = Y = O$ and that the points H, O', O are collinear.

Alternate Solution: Deduction of the fact that both of the quadrilaterals $HBM'M$ and $HCN'N$ are rhombuses is carried out in the same way as in the preceding proof.

We then see that the point M is located in a symmetric position with the point B with respect to the line CH, we conclude that we have $\angle CMB = \beta$. Similarly, we have $\angle CNB = \gamma$. If we now put $x = \angle AHO'$, then we get

$$\angle O' = \beta - \alpha - x, \;\; \angle MNH = 90^\circ - \beta - \alpha + x,$$

from which it follows that

$$\angle ANM = 180^\circ - \angle MNH - (90^\circ - \alpha) = \beta - x.$$

Similarly, we get

$$\angle NMA = \gamma + x.$$

Using the laws of sines, we then get

$$\frac{\sin(\gamma+x)}{\sin(\beta-x)} = \frac{AN}{AM} = \frac{AC}{AM}\cdot\frac{AB}{AC}\cdot\frac{AN}{AB}$$
$$= \frac{\sin\beta}{\sin(\beta-\alpha)}\cdot\frac{\sin\gamma}{\sin\beta}\cdot\frac{\sin(\gamma-\alpha)}{\sin\gamma} = \frac{\sin(\gamma-\alpha)}{\sin(\beta-\alpha)}.$$

On the other hand, if we let $y = \angle AHO$, we then get

$$\angle OHB = 180^\circ - \gamma - y,\ \ \angle CHO = 180^\circ - \beta + y,$$

and since

$$\angle HBO = \gamma - \alpha,\ \ \angle OCH = \beta - \alpha,$$

using the laws of sines and observing that $OB = OC$, we get

$$\frac{\sin(\gamma-\alpha)}{\sin(\beta-\alpha)} = \frac{\sin\angle HBO}{\sin\angle OCH} = \frac{\sin(180^\circ-\gamma-y)\cdot\frac{OH}{OB}}{\sin(180^\circ-\beta+y)\cdot\frac{OH}{OC}}$$
$$= \frac{\sin(180^\circ-\gamma-y)}{\sin(180^\circ-\beta+y)} = \frac{\sin(\gamma+y)}{\sin(\beta-y)}.$$

We then get $\sin(\gamma+x)\sin(\beta-y) = \sin(\beta-x)\sin(\gamma+y)$. Expanding both sides of the last identity by using the addition formula for the sine function and after factoring and using again the addition formula we obtain that $\sin(x-y)\sin(\beta+\gamma) = 0$. This implies that $x-y$ must be an integral multiple of 180°, and hence we conclude that H, O, O' are collinear.

Problem 5. Find all functions f from the set $\mathbf{R}$ of real numbers into $\mathbf{R}$ which satisfy for all $x, y, z \in \mathbf{R}$ the identity

$$f(f(x)+f(y)+f(z)) = f(f(x)-f(y)) + f(2xy+f(z)) + 2f(xz-yz).$$

Solution: It is clear that if f is a constant function which satisfies the given equation, then the constant must be 0. Conversely, $f(x) = 0$ clearly satisfies the given equation, so, the identically 0 function is a solution. In the sequel, we consider the case where f is not a constant function.

Let $t \in \mathbf{R}$ and substitute $(x,y,z) = (t,0,0)$ and $(x,y,z) = (0,t,0)$ into the given functional equation. Then, we obtain, respectively,

$$f(f(t)+2f(0)) = f(f(t)-f(0)) + f(f(0)) + 2f(0),$$
$$f(f(t)+2f(0)) = f(f(0)-f(t)) + f(f(0)) + 2f(0),$$

from which we conclude that $f(f(t)-f(0)) = f(f(0)-f(t))$ holds for all $t \in \mathbf{R}$. Now, suppose for some pair u_1, u_2, $f(u_1) = f(u_2)$ is satisfied. Then by substituting $(x,y,z) = (s,0,u_1)$ and $(x,y,z) = (s,0,u_2)$ into the functional equation and comparing the resulting identities, we can easily conclude that

$$f(su_1) = f(su_2) \qquad (*)$$

holds for all $s \in \mathbf{R}$. Since f is not a constant function there exists an s_0 such that $f(s_0) - f(0) \neq 0$. If we put $u_1 = f(s_0) - f(0), u_2 = -u_1$, then $f(u_1) = f(u_2)$, so we have by $(*)$

$$f(su_1) = f(su_2) = f(-su_1)$$

for all $s \in \mathbf{R}$. Since $u_1 \neq 0$, we conclude that

$$f(x) = f(-x)$$

holds for all $x \in \mathbf{R}$.

Next, if $f(u) = f(0)$ for some $u \neq 0$, then by $(*)$, we have $f(su) = f(s0) = f(0)$ for all s, which implies that f is a constant function, contradicting our assumption. Therefore, we must have $f(s) \neq f(0)$ whenever $s \neq 0$.

We will now show that if $f(x) = f(y)$ holds, then either $x = y$ or $x = -y$ must hold. Suppose on the contrary that $f(x_0) = f(y_0)$ holds for some pair of non-zero numbers x_0, y_0 for which $x_0 \neq y_0, x_0 \neq -y_0$. Since $f(-y_0) = f(y_0)$, we may assume, by replacing y_0 by $-y_0$ if necessary, that x_0 and y_0 have the same sign. In view of $(*)$, we see that $f(sx_0) = f(sy_0)$ holds for all s, and therefore, there exists some $r > 0, r \neq 1$ such that

$$f(x) = f(rx)$$

holds for all x. Replacing x by rx and y by ry in the given functional equation, we obtain

$$f(f(rx) + f(ry) + f(z)) = f(f(rx) - f(ry)) + f(2r^2xy + f(z)) + 2f(r(x-y)z) \qquad \text{(i)},$$

and replacing x by r^2x in the functional equation, we get

$$f(f(r^2x) + f(y) + f(z)) = f(f(r^2x) - f(y)) + f(2r^2xy + f(z)) + 2f((r^2x - y)z) \quad \text{(ii)}.$$

Since $f(rx) = f(x)$ holds for all $x \in \mathbf{R}$, we see that except for the last term on the right-hand side, all the corresponding terms appearing in the identities (i) and (ii) above are equal, and hence we conclude that

$$f(r(x-y)z) = f((r^2x - y)z)) \qquad \text{(iii)}$$

must hold for arbitrary choice of $x, y, z \in \mathbf{R}$. For arbitrarily fixed pair $u, v \in \mathbf{R}$, substitute $(x, y, z) = (\frac{v-u}{r^2-1}, \frac{v-r^2u}{r^2-1}, 1)$ into the identity (iii). Then we obtain $f(v) = f(ru) = f(u)$, since $x - y = u, r^2x - y = v, z = 1$. But this implies that the function f is a constant, contradicting our assumption. Thus we conclude that if $f(x) = f(y)$ then either $x = y$ or $x = -y$ must hold.

By substituting $z = 0$ in the functional equation, we get

$$f(f(x) + f(y) + f(0)) = f(f(x) - f(y) + f(0)) = f((f(x) - f(y)) + f(2xy + f(0)) + 2f(0).$$

Changing y to $-y$ in the identity above and using the fact that $f(y) = f(-y)$, we see that all the terms except the second term on the right-hand side in the identity above remain the same. Thus we conclude that $f(2xy + f(0)) = f(-2xy + f(0))$, from which we get either $2xy + f(0) = -2xy + f(0)$ or $2xy + f(0) = 2xy - f(0)$ for all $x, y \in \mathbf{R}$. The first of these alternatives says that $4xy = 0$, which is impossible if $xy \neq 0$. Therefore the second alternative must be valid and we get that $f(0) = 0$.

Finally, let us show that if f satisfies the given functional equation and is not a constant function, then $f(x) = x^2$. Let $x = y$ in the functional equation, then since $f(0) = 0$, we get

$$f(2f(x) + f(z)) = f(2x^2 + f(z)),$$

from which we conclude that either $2f(x) + f(z) = 2x^2 + f(z)$ or $2f(x) + f(z) = -2x^2 - f(z)$ must hold. Suppose there exists x_0 for which $f(x_0) \neq x_0^2$, then from the second alternative, we see that $f(z) = -f(x_0) - x_0^2$ must hold for all z, which means that f must be a constant function, contrary to our assumption. Therefore, the first alternative above must hold, and we have $f(x) = x^2$ for all x, establishing our claim.

It is easy to check that $f(x) = x^2$ does satisfy the given functional equation, so we conclude that $f(x) = 0$ and $f(x) = x^2$ are the only functions that satisfy the requirement.