Problem 1. Let $ABC$ be an acute triangle with altitudes $AD, BE$ and $CF$, and let $O$ be the center of its circumcircle. Show that the segments $OA, OF, OB, OD, OC, OE$ dissect the triangle $ABC$ into three pairs of triangles that have equal areas.

Solution. Let $M$ and $N$ be midpoints of sides $BC$ and $AC$, respectively. Notice that $\angle MOC = \frac{1}{2} \angle BOC = \angle EAB$, $\angle OMC = 90^\circ = \angle AEB$, so triangles $OMC$ and $AEB$ are similar and we get $\frac{OM}{AE} = \frac{OC}{AB}$. For triangles $ONA$ and $BDA$ we also have $\frac{ON}{BD} = \frac{OA}{BA}$.

Then $OM \cdot AE = ON \cdot BD$ or $BD \cdot OM = AE \cdot ON$.

Denote by $S(\Phi)$ the area of the figure $\Phi$. So, we see that $S(OBD) = \frac{1}{2} \cdot BD \cdot OM = \frac{1}{2} \cdot AE \cdot ON = S(OAE)$. Analogously, $S(OCD) = S(OAF)$ and $S(OCE) = S(OBF)$.

Alternative solution. Let $R$ be the circumradius of triangle $ABC$, and as usual write $A, B, C$ for angles $\angle CAB, \angle ABC, \angle BCA$ respectively, and $a, b, c$ for sides $BC, CA, AB$ respectively. Then the area of triangle $OCD$ is

$$S(OCD) = \frac{1}{2} \cdot OC \cdot CD \cdot \sin(\angle OCD) = \frac{1}{2} R \cdot CD \cdot \sin(\angle OCD).$$

Now $CD = b \cos C$, and

$$\angle OCD = \frac{180^\circ - 2A}{2} = 90^\circ - A$$

(since triangle $OBC$ is isosceles, and $\angle BOC = 2A$). So

$$S(OCD) = \frac{1}{2} R b \cos C \sin(90^\circ - A) = \frac{1}{2} R b \cos C \cos A.$$

A similar calculation gives

$$S(OAF) = \frac{1}{2} OA \cdot AF \cdot \sin(\angle OAF) = \frac{1}{2} R (b \cos A) \sin(90^\circ - C) = \frac{1}{2} R b \cos A \cos C,$$

so $OCD$ and $OAF$ have the same area. In the same way we find that $OBD$ and $OAE$ have the same area, as do $OCE$ and $OBF$.

Problem 2. Determine all positive integers $n$ for which $\frac{n^2+1}{\lceil \sqrt{n} \rceil^2 + 2}$ is an integer. Here $[r]$ denotes the greatest integer less than or equal to $r$.

Solution. We will show that there are no positive integers $n$ satisfying the condition of the problem.

Let $m = \lceil \sqrt{n} \rceil$ and $a = n - m^2$. We have $m \geq 1$ since $n \geq 1$. From $n^2+1 = (m^2+a)^2+1 \equiv (a-2)^2 + 1 \pmod{m^2+2}$, it follows that the condition of the problem is equivalent to the fact that $(a-2)^2 + 1$ is divisible by $m^2+2$. Since we have

$$0 < (a-2)^2 + 1 \leq \max\{2^2, (2m-2)^2\} + 1 \leq 4m^2 + 1 < 4(m^2 + 2),$$

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we see that \((a - 2)^2 + 1 = k(m^2 + 2)\) must hold with \(k = 1, 2\) or 3. We will show that none of these can occur.

**Case 1.** When \(k = 1\). We get \((a - 2)^2 - m^2 = 1\), and this implies that \(a - 2 = \pm 1\), \(m = 0\) must hold, but this contradicts with fact \(m \geq 1\).

**Case 2.** When \(k = 2\). We have \((a - 2)^2 + 1 = 2(m^2 + 2)\) in this case, but any perfect square is congruent to 0, 1, 4 mod 8, and therefore, we have \((a - 2)^2 + 1 \equiv 1, 2, 5 \pmod{8}\), while \(2(m^2 + 2) \equiv 4, 6 \pmod{8}\). Thus, this case cannot occur either.

**Case 3.** When \(k = 3\). We have \((a - 2)^2 + 1 = 3(m^2 + 2)\) in this case. Since any perfect square is congruent to 0 or 1 mod 3, we have \((a - 2)^2 + 1 \equiv 1, 2 \pmod{3}\), while \(3(m^2 + 2) \equiv 0 \pmod{3}\), which shows that this case cannot occur either.

**Problem 3.** For \(2k\) real numbers \(a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k\) define the sequence of numbers \(X_n\) by

\[
X_n = \sum_{i=1}^{k} [a_i n + b_i] \quad (n = 1, 2, \ldots).
\]

If the sequence \(X_n\) forms an arithmetic progression, show that \(\sum_{i=1}^{k} a_i\) must be an integer. Here \([r]\) denotes the greatest integer less than or equal to \(r\).

**Solution.** Let us write \(A = \sum_{i=1}^{k} a_i\) and \(B = \sum_{i=1}^{k} b_i\). Summing the corresponding terms of the following inequalities over \(i\),

\[
a_i n + b_i - 1 < [a_i n + b_i] \leq a_i n + b_i;
\]

we obtain \(An + B - k < X_n < An + B\). Now suppose that \(\{X_n\}\) is an arithmetic progression with the common difference \(d\), then we have \(nd = X_{n+1} - X_1\) and \(A + B - k < X_1 \leq A + B\) Combining with the inequalities obtained above, we get

\[
A(n + 1) + B - k < nd + X_1 < A(n + 1) + B;
\]

or

\[
An - k \leq An + (A + B - X_1) - k < nd < An + (A + B - X_1) < An + k,
\]

from which we conclude that \(|A - d| < \frac{k}{n}\) must hold. Since this inequality holds for any positive integer \(n\), we must have \(A = d\). Since \(\{X_n\}\) is a sequence of integers, \(d\) must be an integer also, and thus we conclude that \(A\) is also an integer.

**Problem 4.** Let \(a\) and \(b\) be positive integers, and let \(A\) and \(B\) be finite sets of integers satisfying:

(i) \(A\) and \(B\) are disjoint;

(ii) if an integer \(i\) belongs either to \(A\) or to \(B\), then \(i + a\) belongs to \(A\) or \(i - b\) belongs to \(B\).

Prove that \(a|A| = b|B|\). (Here \(|X|\) denotes the number of elements in the set \(X\).)

**Solution.** Let \(A^* = \{n - a : n \in A\}\) and \(B^* = \{n + b : n \in B\}\). Then, by (ii), \(A \cup B \subseteq A^* \cup B^*\) and by (i),

\[
|A \cup B| \leq |A^* \cup B^*| \leq |A^*| + |B^*| = |A| + |B| = |A \cup B|.
\] (1)
Thus, $A \cup B = A^* \cup B^*$ and $A^*$ and $B^*$ have no element in common. For each finite set $X$ of integers, let $\sum(X) = \sum_{x \in X} x$. Then
\[
\sum(A) + \sum(B) = \sum(A \cup B) = \sum(A^* \cup B^*) = \sum(A^*) + \sum(B^*)
= \sum(A) - a|A| + \sum(B) + b|B|,
\]
which implies $a|A| = b|B|$.

**Alternative solution.** Let us construct a directed graph whose vertices are labelled by the members of $A \cup B$ and such that there is an edge from $i$ to $j$ iff $j \in A$ and $j = i + a$ or $j \in B$ and $j = i - b$. From (ii), each vertex has out-degree $\geq 1$ and, from (i), each vertex has in-degree $\leq 1$. Since the sum of the out-degrees equals the sum of the in-degrees, each vertex has in-degree and out-degree equal to 1. This is only possible if the graph is the union of disjoint cycles, say $G_1, G_2, \ldots, G_n$. Let $|A_k|$ be the number of elements of $A$ in $G_k$ and $|B_k|$ be the number of elements of $B$ in $G_k$. The cycle $G_k$ will involve increasing vertex labels by $a$ a total of $|A_k|$ times and decreasing them by $b$ a total of $|B_k|$ times. Since it is a cycle, we have $a|A_k| = b|B_k|$. Summing over all cycles gives the result.

**Problem 5.** Let $ABCD$ be a quadrilateral inscribed in a circle $\omega$, and let $P$ be a point on the extension of $AC$ such that $PB$ and $PD$ are tangent to $\omega$. The tangent at $C$ intersects $PD$ at $Q$ and the line $AD$ at $R$. Let $E$ be the second point of intersection between $AQ$ and $\omega$. Prove that $B, E, R$ are collinear.

**Solution.** To show $B, E, R$ are collinear, it is equivalent to show the lines $AD, BE, CQ$ are concurrent. Let $CQ$ intersect $AD$ at $R$ and $BE$ intersect $AD$ at $R'$. We shall show $RD/RA = R'D/R'A$ so that $R = R'$.

Since $\triangle PAB$ is similar to $\triangle PDC$ and $\triangle PAB$ is similar to $\triangle PBC$, we have $AD/DC = PA/PD = PA/PB = AB/BC$. Hence, $AB \cdot DC = BC \cdot AD$. By Ptolemy’s theorem, $AB \cdot DC = BC \cdot AD = \frac{1}{2}CA \cdot DB$. Similarly $CA \cdot ED = CE \cdot AD = \frac{1}{2}AE \cdot DC$.

Thus
\[
\frac{DB}{AB} = \frac{2DC}{CA},
\]
and
\[
\frac{DC}{CA} = \frac{2ED}{AE}.
\]
Since the triangles $RDC$ and $RCA$ are similar, we have $\frac{RD}{RC} = \frac{DC}{CA} = \frac{RC}{RA}$. Thus using (4)

$$\frac{RD}{RA} = \frac{RD \cdot RA}{RA^2} = \left(\frac{RC}{RA}\right)^2 = \left(\frac{DC}{CA}\right)^2 = \left(\frac{2ED}{AE}\right)^2. \quad (5)$$

Using the similar triangles $ABR'$ and $EDR'$, we have $R'D/R'B = ED/AB$. Using the similar triangles $DBR'$ and $EAR'$ we have $R'A/R'B = EA/DB$. Thus using (3) and (4),

$$\frac{R'D}{R'A} = \frac{ED \cdot DB}{EA \cdot AB} = \left(\frac{2ED}{AE}\right)^2. \quad (6)$$

It follows from (5) and (6) that $R = R'$. 