Problem 1. For a positive integer $m$ denote by $S(m)$ and $P(m)$ the sum and product, respectively, of the digits of $m$. Show that for each positive integer $n$, there exist positive integers $a_1, a_2, \ldots, a_n$ satisfying the following conditions:

$$S(a_1) < S(a_2) < \cdots < S(a_n)$$ $$S(a_i) = P(a_{i+1}) \quad (i = 1, 2, \ldots, n).$$

(We let $a_{n+1} = a_1$.)

(Proposed by the Problem Committee of the Japan Mathematical Olympiad Foundation)

Problem 2. Let $S = \{1, 2, \ldots, 2014\}$. For each non-empty subset $T \subseteq S$, one of its members is chosen as its representative. Find the number of ways to assign representatives to all non-empty subsets of $S$ so that if a subset $D \subseteq S$ is a disjoint union of non-empty subsets $A, B, C \subseteq S$, then the representative of $D$ is also the representative of at least one of $A, B, C$.

(Proposed by Warut Suksompong, Thailand)

Problem 3. Find all positive integers $n$ such that for any integer $k$ there exists an integer $a$ for which $a^3 + a - k$ is divisible by $n$.

(Proposed by Warut Suksompong, Thailand)

Problem 4. Let $n$ and $b$ be positive integers. We say $n$ is $b$-discerning if there exists a set consisting of $n$ different positive integers less than $b$ that has no two different subsets $U$ and $V$ such that the sum of all elements in $U$ equals the sum of all elements in $V$.

(a) Prove that 8 is a 100-discerning.

(b) Prove that 9 is not 100-discerning.

(Proposed by the Senior Problems Committee of the Australian Mathematical Olympiad Committee)

Problem 5. Circles $\omega$ and $\Omega$ meet at points $A$ and $B$. Let $M$ be the midpoint of the arc $AB$ of circle $\omega$ ($M$ lies inside $\Omega$). A chord $MP$ of circle $\omega$ intersects $\Omega$ at $Q$ ($Q$ lies inside $\omega$). Let $\ell_P$ be the tangent line to $\omega$ at $P$, and let $\ell_Q$ be the tangent line to $\Omega$ at $Q$. Prove that the circumcircle of the triangle formed by the lines $\ell_P$, $\ell_Q$, and $AB$ is tangent to $\Omega$.

(Proposed by Ilya Bogdanov, Russia and Medeubek Kungozhin, Kazakhstan)