XXVII Asian Pacific Mathematics Olympiad

Time allowed: 4 hours *Each problem if worth 7 points*

Problem 1. Let ABC be a triangle, and let D be a point on side BC. A line through D intersects side AB at X and ray AC at Y. The circumcircle of triangle BXD intersects the circumcircle ω of triangle ABC again at point $Z \neq B$. The lines ZD and ZY intersect ω again at V and W, respectively. Prove that $AB = VW$.

Proposed by Warut Suksompong, Thailand

Problem 2. Let $S = \{2, 3, 4, \ldots\}$ denote the set of integers that are greater than or equal to 2. Does there exist a function $f : S \to S$ such that

$$f(a)f(b) = f(a^2b^2) \text{ for all } a, b \in S \text{ with } a \neq b?$$

Proposed by Angelo Di Pasquale, Australia

Problem 3. A sequence of real numbers $a_0, a_1, \ldots$ is said to be *good* if the following three conditions hold.

(i) The value of a_0 is a positive integer.

(ii) For each non-negative integer i we have $a_{i+1} = 2a_i + 1$ or $a_{i+1} = \dfrac{a_i}{a_i + 2}$.

(iii) There exists a positive integer k such that $a_k = 2014$.

Find the smallest positive integer n such that there exists a good sequence $a_0, a_1, \ldots$ of real numbers with the property that $a_n = 2014$.

Proposed by Wang Wei Hua, Hong Kong

Problem 4. Let n be a positive integer. Consider $2n$ distinct lines on the plane, no two of which are parallel. Of the $2n$ lines, n are colored blue, the other n are colored red. Let $\mathcal{B}$ be the set of all points on the plane that lie on at least one blue line, and $\mathcal{R}$ the set of all points on the plane that lie on at least one red line. Prove that there exists a circle that intersects $\mathcal{B}$ in exactly $2n - 1$ points, and also intersects $\mathcal{R}$ in exactly $2n - 1$ points.

Proposed by Pakawut Jiradilok and Warut Suksompong, Thailand

Problem 5. Determine all sequences $a_0, a_1, a_2, \ldots$ of positive integers with $a_0 \geq 2015$ such that for all integers $n \geq 1$:

(i) a_{n+2} is divisible by a_n;

(ii) $|s_{n+1} - (n+1)a_n| = 1$, where $s_{n+1} = a_{n+1} - a_n + a_{n-1} - \cdots + (-1)^{n+1}a_0$.

Proposed by Pakawut Jiradilok and Warut Suksompong, Thailand