Solutions of APMO 2017

Problem 1. We call a 5-tuple of integers *arrangeable* if its elements can be labeled a, b, c, d, e in some order so that $a - b + c - d + e = 29$. Determine all 2017-tuples of integers $n_1, n_2, \dots, n_{2017}$ such that if we place them in a circle in clockwise order, then any 5-tuple of numbers in consecutive positions on the circle is arrangeable.

Answer: $n_1 = \cdots = n_{2017} = 29$.

Solution. A valid 2017-tuple is $n_1 = \cdots = n_{2017} = 29$. We will show that it is the only solution.

We first replace each number n_i in the circle by $m_i := n_i - 29$. Since the condition $a - b + c - d + e = 29$ can be rewritten as $(a-29) - (b-29) + (c-29) - (d-29) + (e-29) = 0$, we have that any five consecutive replaced integers in the circle can be labeled a, b, c, d, e in such a way that $a - b + c - d + e = 0$. We claim that this is possible only when all of the m_i's are 0 (and thus all of the original n_i's are 29).

We work with indexes modulo 2017. Notice that for every i, m_i and m_{i+5} have the same parity. Indeed, this follows from $m_i \equiv m_{i+1} + m_{i+2} + m_{i+3} + m_{i+4} \equiv m_{i+5} \pmod 2$. Since $\gcd(5, 2017) = 1$, this implies that all m_i's are of the same parity. Since $m_1 + m_2 + m_3 + m_4 + m_5$ is even, all m_i's must be even as well.

Suppose for the sake of contradiction that not all m_i's are zero. Then our condition still holds when we divide each number in the circle by 2. However, by performing repeated divisions, we eventually reach a point where some m_i is odd. This is a contradiction.

$\square$

Problem 2. Let ABC be a triangle with $AB < AC$. Let D be the intersection point of the internal bisector of angle BAC and the circumcircle of ABC. Let Z be the intersection point of the perpendicular bisector of AC with the external bisector of angle $\angle BAC$. Prove that the midpoint of the segment AB lies on the circumcircle of triangle ADZ.

Solution 1. Let N be the midpoint of AC. Let M be the intersection point of the circumcircle of triangle ADZ and the segment AB. We will show that M is the midpoint of AB. To do this, let D' the reflection of D with respect to M. It suffices to show that $ADBD'$ is a parallelogram.

The internal and external bisectors of an angle in a triangle are perpendicular. This implies that ZD is a diameter of the circumcircle of AZD and thus $\angle ZMD = 90°$. This means that ZM is the perpendicular bisector of $D'D$ and thus $ZD' = ZD$. By construction, Z is in the perpendicular bisector of AC and thus $ZA = ZC$.

Now, let α be the angle $\angle BAD = \angle DAC$. In the cyclic quadrilateral $AZDM$ we get $\angle MZD = \angle MAD = \alpha$, and thus $\angle D'ZD = 2\alpha$. By angle chasing we get

$$\angle AZN = 90° - \angle ZAN = \angle DAC = \alpha,$$

which implies that $\angle AZC = 2\alpha$. Therefore,

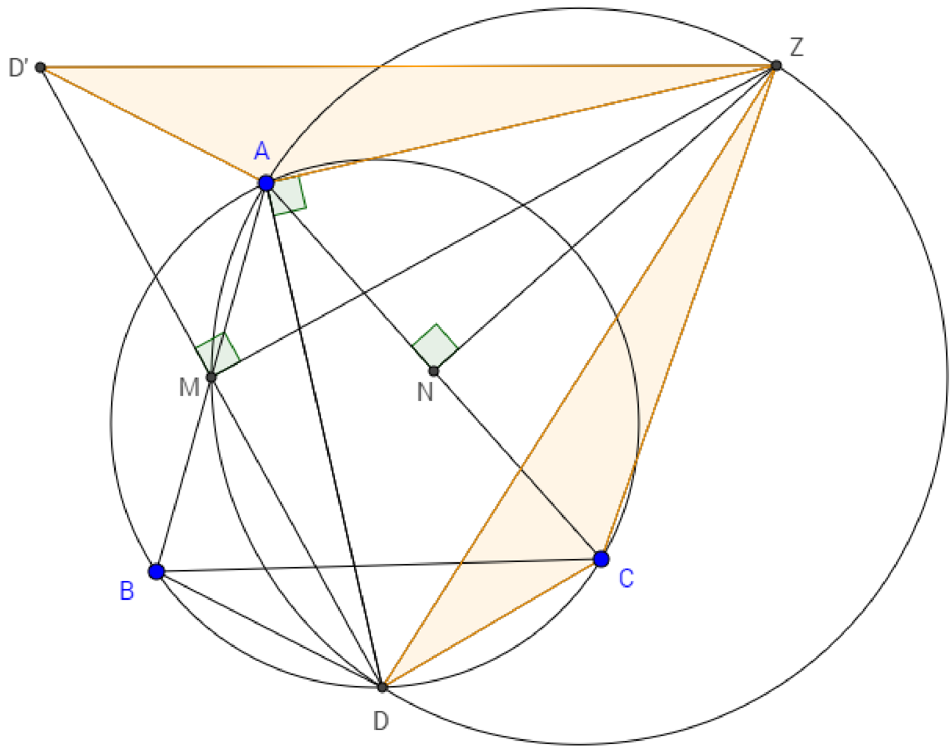
$$\angle D'ZA = \angle D'ZD - \angle AZD = 2\alpha - \angle AZD = \angle AZC - \angle AZD = \angle DZC.$$

Combining $\angle D'ZA = \angle DZC$, $ZD' = ZD$ and $ZA = ZC$, we obtain by the SAS criterion that the triangles $D'ZA$ and DZC are congruent. In particular, $D'A = DC$ and $\angle D'AZ = \angle DCZ$. From here $DB = DC = D'A$.

Finally, let $\beta = \angle ABC = \angle ADC$. We get the first of the following equalities by the sum of angles around point A and the second one by the sum of internal angles of quadrilateral $AZCD$

$$360^\circ = \angle D'AZ + \angle ZAD + \angle DAB + \angle BAD' = \angle D'AZ + 90^\circ + \alpha + \angle BAD'$$
$$360^\circ = \angle DCZ + \angle ZAD + \angle CZA + \angle ADC = \angle DCZ + 90^\circ + 2\alpha + \beta.$$

By canceling equal terms we conclude that $\angle BAD' = \alpha + \beta$. Also, $\angle ABD = \alpha + \beta$. Therefore, the segments $D'A$ and DB are parallel and have the same length. We conclude that $ADBD'$ is a parallelogram. As the diagonals of a parallelogram intersect at their midpoints, we obtain that M is the midpoint of AB as desired.



□

Variant of solution. The solution above is indirect in the sense that it assumes that M is in the circumcircle of AZD and then shows that M is the midpoint of AB. We point out that the same ideas in the solution can be used to give a direct solution. Here we present a sketch on how to proceed in this manner.

Now we know that M is the midpoint of the side AB. We construct the point D' in the same way. Now we have directly that $ADBD'$ is a parallelogram and thus $D'A = DB = DC$. By construction $ZA = ZC$. Also, the two sums of angles equal to 360° in the previous solution let us conclude that $\angle D'AZ = \angle DCZ$. Once again, we use (differently) the SAS criterion and obtain that the triangles $D'AZ$ and DCZ are congruent. Thus, $D'Z = DZ$.

We finish the problem by noting that ZM is a median of the isosceles triangle $D'ZD$, so it is also a perpendicular bisector. This shows that $\angle DMZ = 90^\circ = \angle DAZ$, and therefore M lies in the circumcircle of DAZ.

Solution 2. We proceed directly. As above, we name

$$\alpha = \angle DAC = \angle AZN = \angle CZN = \angle DCB.$$

Let L be the midpoint of the segment BC. Since M and N are midpoints of AB and AC, we have that $MN = CL$ and that the segment MN is parallel to BC. Thus, $\angle ANM = \angle ACB$.

Therefore, $\angle ZNM = \angle ACB + 90°$. Also, $\angle DCZ = \angle DCB + \angle ACB + \angle ZCA = \angle ACB + 90°$. We conclude that $\angle ZNM = \angle DCZ$.

Now, the triangles ZNC and CLD are similar since $\angle DCL = 90° = \angle CNZ$ and $\angle DCL = \alpha = \angle CZN$. We use this fact to obtain the following chain of equalities

$$\frac{CD}{MN} = \frac{CD}{CL} = \frac{CZ}{ZN}.$$

We combine the equality above with $\angle ZNM = \angle DCZ$ to conclude that the triangles ZNM and ZCD are similar. In particular, $\angle MZN = \angle DZC$ and $\frac{ZM}{ZN} = \frac{ZD}{ZC}$. Since we also have $\angle MZD = \angle MZN + \angle NZD = \angle DZC + \angle NZD = \angle NZC$, we conclude that the triangles MZD and NZC are similar. This yields that $\angle ZMD = 90°$ and therefore M is in the circumcircle of triangle DAZ.

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Solution 3. Let m be the perpendicular bisector of AD; thus m passes through the center O of the circumcircle of triangle ABC. Since AD is the internal angle bisector of A and OM and ON are perpendicular to AB and AC respectively, we obtain that OM and ON form equal angles with AD. This implies that they are symmetric with respect to M.

Therefore, the line MO passes through the point Z' symmetrical to Z with respect to m. Since $\angle DAZ = 90°$, then also $\angle ADZ' = 90°$. Moreover, since $AZ = DZ'$, we have that $\angle AZZ' = \angle DZ'Z = 90°$, so $AZZ'D$ is a rectangle. Since $\angle AMZ' = \angle AMO = 90°$, we conclude that M is in the circle with diameter AZ', which is the circumcircle of the rectangle. Thus M lies on the circumcircle of the triangle ADZ.

Problem 3. Let $A(n)$ denote the number of sequences $a_1 \geq a_2 \geq \ldots \geq a_k$ of positive integers for which $a_1 + \cdots + a_k = n$ and each $a_i + 1$ is a power of two ($i = 1, 2, \ldots, k$). Let $B(n)$ denote the number of sequences $b_1 \geq b_2 \geq \ldots \geq b_m$ of positive integers for which $b_1 + \cdots + b_m = n$ and each inequality $b_j \geq 2b_{j+1}$ holds ($j = 1, 2, \ldots, m-1$).

Prove that $A(n) = B(n)$ for every positive integer n.

Solution. We say that a sequence $a_1 \geq a_2 \geq \ldots \geq a_k$ of positive integers has *type A* if $a_i + 1$ is a power of two for $i = 1, 2, \ldots, k$. We say that a sequence $b_1 \geq b_2 \geq \ldots \geq b_m$ of positive integer has *type B* if $b_j \geq 2b_{j+1}$ for $j = 1, 2, \ldots, m-1$.

Recall that the binary representation of a positive integer expresses it as a sum of distinct powers of two in a unique way. Furthermore, we have the following formula for every positive integer N

$$2^N - 1 = 2^{N-1} + 2^{N-2} + \cdots + 2^1 + 2^0.$$

Given a sequence $a_1 \geq a_2 \geq \ldots \geq a_k$ of type A, use the preceding formula to express each term as a sum of powers of two. Write these powers of two in left-aligned rows, in decreasing order of size. By construction, a_i is the sum of the numbers in the ith row. For example, we obtain the following array when we start with the type A sequence $15, 15, 7, 3, 3, 3, 1$.

8	4	2	1
8	4	2	1
4	2	1	
2	1		
2	1		
2	1		
1			
27	13	5	2

Define the sequence $b_1, b_2, \ldots, b_n$ by setting b_j to be the sum of the numbers in the jth column of the array. For example, we obtain the sequence $27, 13, 5, 2$ from the array above. We now show that this new sequence has type B. This is clear from the fact that each column in the array contains at least as many entries as the column to the right of it and that each number larger than 1 in the array is twice the number to the right of it. Furthermore, it is clear that $a_1 + a_2 + \cdots + a_k = b_1 + b_2 + \cdots + b_m$, since both are equal to the sum of all the entries in the array.

We now show that we can do this operation backwards. Suppose that we are given a type B sequence $b_1 \geq b_2 \geq \ldots \geq b_m$. We construct an array inductively as follows:

- We fill b_m left-aligned rows with the numbers $2^{m-1}, 2^{m-2}, \ldots, 2^1, 2^0$.
- Then we fill $b_{m-1} - 2b_m$ left aligned rows with the numbers $2^{m-2}, 2^{m-3}, \ldots, 2^1, 2^0$.
- Then we fill $b_{m-2} - 2b_{m-1}$ left aligned rows with the numbers $2^{m-3}, 2^{m-4}, \ldots, 2^1, 2^0$, and so on.
- In the last step we fill $b_1 - 2b_2$ left-aligned rows with the number 1.

For example, if we start with the type B sequence $27, 13, 5, 2$, we obtain once again the array above. We define the sequence $a_1, a_2, \ldots, a_k$ by setting a_i to be the sum of the numbers in the ith row of the array. By construction, this sequence has type A. Furthermore, it is clear that $a_1 + \cdots + a_k = b_1 + \cdots + b_m$, since once again both sums are equal to the sum of all the entries in the array.

We have defined an operation that starts with a sequence of type A, produces an array whose row sums are given by the sequence, and outputs a sequence of type B corresponding to the column sums. We have also defined an operation that starts with a sequence of type B, produces an array whose column sums are given by the sequence, and outputs a sequence of type A corresponding to the row sums. The arrays produced in both cases comprise left-aligned rows of the form $2^{N-1}, 2^{N-2}, \ldots, 2^1, 2^0$, with non-increasing lengths. Let us refer to arrays obeying these properties as *marvelous*.

To show that these two operations are inverses of each other, it then suffices to prove that marvelous arrays are uniquely defined by either their row sums or their column sums. The former is obviously true and the latter arises from the observation that each step in the above inductive algorithm was forced in order to create a marvelous array with the prescribed column sums.

Thus, we have produced a bijection between the sequences of type A with sum n and the sequences of type B with sum n. So we can conclude that $A(n) = B(n)$ for every positive integer n.

$\square$

Remark The solution above provides a bijection between type A and type B sequences via an algorithm. There are alternative ways to provide such a bijection. For example, given the numbers $a_1 \geq \ldots \geq a_k$ we may define the b_i's as

$$b_j = \sum_i \left\lfloor \frac{a_i + 1}{2^j} \right\rfloor.$$

Conversely, given the numbers $b_1 \geq \ldots \geq b_m$, one may define the a_i's by taking, as in the solution, b_m numbers equal to $2^m - 1$, $b_{m-1} - 2b_m$ numbers equal to $2^{m-1} - 1, \ldots$, and $b_1 - 2b_2$ numbers equal to $2^1 - 1$. One now needs to verify that these maps are mutually inverse.

Problem 4. Call a rational number r *powerful* if r can be expressed in the form $\frac{p^k}{q}$ for some relatively prime positive integers p, q and some integer $k > 1$. Let a, b, c be positive rational numbers such that $abc = 1$. Suppose there exist positive integers x, y, z such that $a^x + b^y + c^z$ is an integer. Prove that a, b, c are all *powerful.*

Solution. Let $a = \frac{a_1}{b_1}, b = \frac{a_2}{b_2}$, where $\gcd(a_1, b_1) = \gcd(a_2, b_2) = 1$. Then $c = \frac{b_1 b_2}{a_1 a_2}$. The condition that $a^x + b^y + c^z$ is an integer becomes

$$\frac{a_1^{x+z} a_2^z b_2^y + a_1^z a_2^{y+z} b_1^x + b_1^{x+z} b_2^{y+z}}{a_1^z a_2^z b_1^x b_2^y} \in \mathbb{Z},$$

which can be restated as

$$a_1^z a_2^z b_1^x b_2^y \mid a_1^{x+z} a_2^z b_2^y + a_1^z a_2^{y+z} b_1^x + b_1^{x+z} b_2^{y+z}. \tag{1}$$

In particular, a_1^z divides the right-hand side. Since it divides the first and second terms in the sum, we conclude that $a_1^z \mid b_1^{x+z} b_2^{y+z}$. Since $\gcd(a_1, b_1) = 1$, we have $a_1^z \mid b_2^{y+z}$.

Let p be a prime that divides a_1. Let $m, n \geq 1$ be integers such that $p^n \| a_1$ (i.e. $p^n | a_1$ but $p^{n+1} \nmid a_1$) and $p^m \| b_2$. The fact that $a_1^z \mid b_2^{y+z}$ implies $nz \leq m(y + z)$. Since $\gcd(a_1, b_1) = \gcd(a_2, b_2) = 1$, we have p does not divide b_1 and does not divide a_2. Thus

$$p^{nz} \| a_1^z a_2^{y+z} b_1^x \quad \text{and} \quad p^{m(y+z)} \| b_1^{x+z} b_2^{y+z}. \tag{2}$$

On the other hand, (1) implies that

$$p^{nz+my} \mid a_1^z a_2^{y+z} b_1^x + b_1^{x+z} b_2^{y+z}. \tag{3}$$

If $nz < m(y + z)$, then (2) gives $p^{nz} \| a_1^z a_2^{y+z} b_1^x + b_1^{x+z} b_2^{y+z}$, which contradicts (3). Thus $nz = m(y + z)$ so n is divisible by $k := \dfrac{y + z}{\gcd(z, y + z)} > 1$. Thus each exponent in the prime decomposition of a_1 must be divisible by k. Hence a_1 is a perfect k-power which means a is powerful. Similarly, b and c are also powerful.

$\square$

Problem 5. Let n be a positive integer. A pair of n-tuples $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ with integer entries is called an *exquisite pair* if

$$|a_1 b_1 + \cdots + a_n b_n| \leq 1.$$

Determine the maximum number of distinct n-tuples with integer entries such that any two of them form an exquisite pair.

Answer: The maximum is $n^2 + n + 1$.

Solution. First, we construct an example with $n^2 + n + 1$ n-tuples, each two of them forming an exquisite pair. In the following list, $*$ represents any number of zeros as long as the total number of entries is n.

- $(*)$
- $(*, 1, *)$
- $(*, -1, *)$
- $(*, 1, *, 1, *)$
- $(*, 1, *, -1, *)$

For example, for $n = 2$ we have the tuples $(0,0)$, $(0,1)$, $(1,0)$, $(0,-1)$, $(-1,0)$, $(1,1)$, $(1,-1)$.

The total number of such tuples is $1+n+n+\binom{n}{2}+\binom{n}{2} = n^2+n+1$. For any two of them, at most two of the products a_ib_i are non-zero. The only case in which two of them are non-zero is when we take a sequence $(*,1,*,1,*)$ and a sequence $(*,1,*,-1,*)$ with zero entries in the same places. But in this case one a_ib_i is 1 and the other -1. This shows that any two of these sequences form an exquisite pair.

Next, we claim that among any n^2+n+2 tuples, some two of them do not form an exquisite pair. We begin with lemma.

Lemma. Given $2n+1$ distinct non-zero n-tuples of real numbers, some two of them $(a_1,\dots,a_n)$ and $(b_1,\dots,b_n)$ satisfy $a_1b_1+\cdots+a_nb_n > 0$.

Proof of Lemma. We proceed by induction. The statement is easy for $n = 1$ since for every three non-zero numbers there are two of them with the same sign. Assume that the statement is true for $n-1$ and consider $2n+1$ tuples with n entries. Since we are working with tuples of real numbers, we claim that we may assume that one of the tuples is $a = (0,0,\dots,0,-1)$. Let us postpone the proof of this claim for the moment.

If one of the remaining tuples b has a negative last entry, then a and b satisfy the desired condition. So we may assume all the remaining tuples has a non-negative last entry. Now, from each tuple remove the last number. If two n-tuples b and c yield the same $(n-1)$-tuple, then

$$b_1c_1+\cdots+b_{n-1}c_{n-1}+b_nc_n = b_1^2+\cdots+b_{n-1}^2+b_nc_n > 0,$$

and we are done. The remaining case is that all the n-tuples yield distinct $(n-1)$-tuples. Then at most one of them is the zero $(n-1)$-tuple, and thus we can use the inductive hypothesis on $2n-1$ of them. So we find b and c for which

$$(b_1c_1+\cdots+b_{n-1}c_{n-1})+b_nc_n > 0+b_nc_n > 0.$$

The only thing that we are left to prove is that in the inductive step we may assume that one of the tuples is $a = (0,0,\dots,0,-1)$. Fix one of the tuples $x = (x_1,\dots,x_n)$. Set a real number φ for which $\tan\varphi = \frac{x_1}{x_2}$. Change each tuple $a = (a_1,a_2,\dots,a_n)$ (including x), to the tuple

$$(a_1\cos\varphi - a_2\sin\varphi, a_1\sin\varphi + a_2\cos\varphi, a_3, a_4, \dots, a_n).$$

A straightforward calculation shows that the first coordinate of the tuple x becomes 0, and that all the expressions of the form $a_1b_1+\cdots+a_nb_n$ are preserved. We may iterate this process until all the entries of x except for the last one are equal to 0. We finish by multiplying all the entries in all the tuples by a suitable constant that makes the last entry of x equal to -1. This preserves the sign of all the expressions of the form $a_1b_1+\cdots+a_nb_n$.

$\square$

We proceed to the proof of our claim. Let A be a set of non-zero tuples among which any two form an exquisite pair. It suffices to prove that $|A| \le n^2+n$. We can write A as a disjoint union of subsets $A_1 \cup A_2 \cup \ldots \cup A_n$, where A_i is the set of tuples in A whose last non-zero entry appears in the ith position. We will show that $|A_i| \le 2i$, which will finish our proof since $2+4+\cdots+2n = n^2+n$.

Proceeding by contradiction, suppose that $|A_i| \ge 2i+1$. If A_i has three or more tuples whose only non-zero entry is in the ith position, then for two of them this entry has the same sign. Since the tuples are different and their entries are integers, this yields two tuples for which $|\sum a_ib_i| \ge 2$, a contradiction. So there are at most two such tuples. We remove them from A_i.

Now, for each of the remaining tuples a, if it has a positive ith coordinate, we keep a as it is. If it has a negative ith coordinate, we replace it with the *opposite tuple* $-a$ with entries with opposite signs. This does not changes the exquisite pairs condition.

After making the necessary changes, we have two cases. The first case is that there are two tuples a and b that have the same first $i-1$ coordinates and thus

$$a_1b_1 + \cdots + a_{i-1}b_{i-1} = a_1^2 + \cdots + a_{i-1}^2 > 0,$$

and thus is at least 1 (the entries are integers). The second case is that no two tuples have the same first $i-1$ coordinates, but then by the Lemma we find two tuples a and b for which

$$a_1b_1 + \cdots + a_{i-1}b_{i-1} \geq 1.$$

In any case, we obtain

$$a_1b_1 + \cdots + a_{i-1}b_{i-1} + a_ib_i \geq 2.$$

This yields a final contradiction to the exquisite pair hypothesis.

□